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Approximate Calculation of Transient Heat Conduction

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Nomenclature

A	= constant in the given boundary heat flux
a	= constant in the given boundary temperature
f	= temperature profile
HBI	= heat-balance integral
k	= thermal conductivity
n	= exponent
Q	= dimensionless heat flux at $x=0$, Eq. (10)
q	= heat flux
T	= temperature
t	= time
x	= space coordinate
α	= thermal diffusivity
β	= a profile parameter
Γ	= Gamma function
δ	= modified thermal penetration depth
Θ	= dimensionless boundary temperature, Eq. (17)
θ	$=T-T_\infty$

Subscripts

∞	= condition at $x=\infty$ or initial condition
0	= condition at $x=0$

Introduction

THE purpose of this Note is to present a simple, yet accurate, method for solving a variety of transient heat-conduction problems. The method represents a further development of the basic ideas used previously in the approximate calculation of skin friction[1,2] and heat transfer[3] associated with boundary-layer flows. It is an integral approach, and its distinguishing feature lies in the use of an integral expression for the boundary flux. In the application to heat-conduction calculations, the method may be considered as a refinement of the heat-balance integral (HBI) method due to Goodman.[4]

In the present Note, the application of the method is illustrated by solving a class of transient heat-conduction problems in which the surface condition (temperature or heat flux) varies as a power of time. The approximate solutions are compared with corresponding exact solutions as well as the solutions by the HBI method. The accuracy and relative merits of the present method are thus made evident. We note that the method is equally applicable to nonlinear problems of heat conduction, e.g., problems involving phase transitions.

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Method of Solution

Consider the standard equation of one-dimensional transient heat conduction in a semi-infinite solid,

$$\frac{\partial \theta}{\partial t}=\alpha\frac{\partial^2\theta}{\partial x^2},\ t>0,\quad 0<x<\infty \tag{1}$$

where, for simplicity, the thermal properties of the solid are assumed constant. An integration of Eq. (1) over the entire region of interest yields the heat-balance integral as

$$\int_0^\infty \frac{\partial\theta}{\partial t}\,\mathrm{d}x=\alpha\left(\frac{\partial\theta}{\partial x}\right)_\infty-\alpha\left(\frac{\partial\theta}{\partial x}\right)_0 \tag{2}$$

which, after assuming $(\partial\theta/\partial x)_\infty=0$, reduces to

$$-\alpha\left(\frac{\partial\theta}{\partial x}\right)_0=\int_0^\infty\frac{\partial\theta}{\partial t}\,\mathrm{d}x \tag{3}$$

In the usual HBI approach,[4] Eq. (3) is the sole equation for the problem once an approximate temperature profile, $\theta=f(x,t)$ containing one profile parameter, is substituted into Eq. (3). However, in the present treatment, Eq. (3) is used as an expression for the boundary heat flux, $(\partial\theta/\partial x)_0$, and a second equation is to be generated for the determination of the profile parameter.

In the present Note, we choose to generate the second equation by an integration of a moment-like equation. Thus, multiplying Eq. (1) by θ and integrating over the entire domain of interest, we get

$$\int_0^\infty\frac{\partial\theta^2}{\partial t}\,\mathrm{d}x=\alpha\left(2\theta\frac{\partial\theta}{\partial x}\right)_\infty-\alpha\left(2\theta\frac{\partial\theta}{\partial x}\right)_0-2\alpha\int_0^\infty\left(\frac{\partial\theta}{\partial x}\right)^2\mathrm{d}x \tag{4}$$

With the usual boundary conditions of $\theta_\infty=(\partial\theta/\partial x)_\infty=0$. Eq. (4) reduces to

$$\int_0^\infty\frac{\partial\theta^2}{\partial t}\,\mathrm{d}x=-2\alpha\theta_0\left(\frac{\partial\theta}{\partial x}\right)_0-2\alpha\int_0^\infty\left(\frac{\partial\theta}{\partial x}\right)^2\mathrm{d}x \tag{5}$$

Equations (3) and (5) form the basis of the solution process when a temperature profile, f, is substituted for θ. For problems where θ_0 is given, the two equations combine to determine the unknowns $(\partial\theta/\partial x)_0$ and the profile parameter. If the surface heat flux is given, a profile containing two parameters will generally be used, and Eqs. (3) and (5) are used to determine the two parameters.

Applications

Power-Law Boundary Temperature

We first consider the case where $\theta_0=at^{n/2}\,(n\geq 0)$ with $\theta_\infty=0$. An exponential profile will be assumed for the temperature, i.e.,

$$f=\mathrm{a}t^{n/2}e^{-x/\delta} \tag{6}$$

which satisfies the essential boundary conditions of $\theta_0=\mathrm{a}t^{n/2}$ and $\theta_\infty=0$. The profile parameter, $\delta(t)$, plays the role of thermal penetration depth but with a slightly modified definition.

Substitution of Eq. (6) into Eqs. (3) and (5) gives the following pair of equations:

$$3\frac{\mathrm{d}\delta}{\mathrm{d}t}+n\frac{\delta}{t}=2\frac{\alpha}{\delta} \tag{7}$$

and

$$-\left(\frac{\partial\theta}{\partial x}\right)_0=\mathrm{a}t^{n/2}\left(\frac{n}{2}\frac{\delta}{t}+\frac{\mathrm{d}\delta}{\mathrm{d}t}\right) \tag{8}$$

Table 1 Boundary heat flux Q for $\theta_0 = at^{n/2}$

n	0	1	2	3	4	5	6	7
Exact	0.5642	0.8862	1.1284	1.3293	1.5045	1.6617	1.8048	1.9386
Present	0.5774	0.8944	1.1339	1.3333	1.5076	1.6641	1.8074	1.9403
HBI	0.7072	1	1.2247	1.4142	1.5811	1.7321	1.8708	2

Table 2 Boundary temperature Θ for $q_0 = at^{n/2}$

n	0	1	2	3	4	5	6	7
Exact	1.1284	0.8862	0.7523	0.6647	0.6018	0.5539	0.5158	0.4847
Present	1.1180	0.8819	0.7500	0.6633	0.6009	0.5533	0.5154	0.4843
Vallerani	1	0.8165	0.7071	0.6325	0.5774	0.5345	0.5000	0.4714

Solutions are easily obtained as

$$\delta/\sqrt{\alpha t} = 2/(3+2n)^{1/2} \tag{9}$$

and

$$Q \equiv q_0\sqrt{\alpha t}/(k\theta_0) = (1+n)/\sqrt{3+2n} \tag{10}$$

where $q_0 = -k(\partial\theta/\partial x)_0$.

Exact solutions to this problem for positive integer values of n can be found from Carslaw and Jaeger.[5]

$$Q = \Gamma(1+\frac{n}{2})/\Gamma(\frac{n+1}{2}) \tag{11}$$

Present solutions are compared with exact solutions in Table 1. The corresponding solutions by the usual HBI method with the same profile are also included. The present results are indeed remarkably accurate, as shown in Table 1. The considerable improvement on the HBI method is also in evidence.

Power-Law Boundary Heat Flux

Consider the problem where $q_0 = -k(\partial\theta/\partial x)_0 = \mathrm{a}t^{n/2}$, ($n \geq 0$). An exponential profile

$$f = \frac{q_0\delta}{k}\beta e^{-x/\delta} \tag{12}$$

is used in the calculation, and it contains two parameters, δ and β. Note that the profile satisfies only the boundary condition of $\theta_\infty = 0$; the boundary flux, $(\partial\theta/\Theta x)_0$, is not to be obtained from $(\partial f/\partial x)_0$.

Equations (3) and (5) reduce, respectively, to

$$\frac{d}{dt}(q_0\delta^2\beta) = \alpha q_0 \tag{13}$$

and

$$\frac{1}{2}\frac{d}{dt}(\beta^2 q_0^2\delta^3) = 2\alpha q_0^2\beta\delta - \beta^2\alpha q_0^2\delta \tag{14}$$

after f is substituted for θ in Eqs. (3) and (5). Solutions are readily found as

$$\beta = (n+\frac{5}{2})/(n+2) = \text{const.} \tag{15}$$

and

$$\delta/\sqrt{\alpha t} = \frac{2}{(2n+5)^{1/2}} \tag{16}$$

The surface temperature θ_0 then follows directly from Eq. (12) as

$$\Theta \equiv \frac{k\theta_0}{q_0\sqrt{\alpha t}} = \frac{(2n+5)^{1/2}}{(n+2)} \tag{17}$$

Again, exact solutions to the problem are available in Carslaw and Jaeger.[5] For integer values of $n \geq -1$, the exact solutions are

$$\Theta \equiv \Gamma(1+\frac{n}{2})/\Gamma(\frac{n}{2}+\frac{3}{2}) \tag{18}$$

The present solutions are summarized in Table 2, and compared with the exact solutions. Results obtained from the usual heat-balance method with the same profile ($\beta = 1$) have recently been reported by Vallerani,[6] and are also included in the table for comparison. The accuracy of the present method continues to be remarkably good as the results in Table 2 indicate.

Concluding Remarks

A basic feature common to both the previous boundary-layer calculations[1-3] and the present heat-conduction calculation is the use of an integral representation for the boundary flux (momentum or heat flux). Unlike the usual integral method in which the boundary flux is obtained from the slope of the assumed profile at the boundary, the present method calculates the flux by considering the momentum or heat balance in the entire region of interest. This basic idea is believed to be responsible for the remarkably good results achieved. It should be noted that the choice of the second equation in the present method is somewhat arbitrary. In fact, the double-integration scheme as used in Refs. 1-3 had been successfully employed earlier[7] in the heat-conduction calculations. Reference 7 contains applications to nonlinear problems involving phase transitions as well as some simple linear problems. Thus, the flexibility with which the basic idea can be used is made evident.

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Determination of Stagnation Chamber Temperature in High-Enthalpy Nozzle Flows

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Introduction

IN high-enthalpy wind-tunnel facilities, the stagnation chamber pressure p_0 and the mass flow rate $\dot{m}$ can be measured directly. The stagnation chamber temperature T_0, which lies typically between 2000 and 5000 K, has to be determined indirectly. This often is done by the simple assumption of inviscid, one-dimensional flow in the subsonic part of the nozzle.[1] For a diatomic gas such as nitrogen, Bachour and Erdtel[2] additionally assumed that the flow was in thermodynamic equilibrium up to the throat and vibrationally frozen thereafter. Carden[3] assumed the flow to be frozen all along the nozzle. The complete neglect of viscous effects in the subsonic part may be questionable in low Reynolds number flow. In particular, the neglect of the wall cooling could lead to substantial error in the T_0 computation. The T_0 calculated from an inviscid flow model in effect replaces an actual nonuniform T_0 profile at the throat by an average uniform one of magnitude smaller than the actual stagnation temperature on the axis at the throat.

Since this computed T_0 generally is used to analyze the flowfield in the uniform core at the nozzle exit from experimental pitot pressure surveys, one may expect significant errors in the calculated test chamber conditions. The purpose of the present note is to suggest an alternative method, based on slender channel analysis,[4] for the calculation of an average stagnation chamber temperature from experimental p_0 and $\dot{m}$. This method takes into account viscous effects, wall cooling, and vibrational relaxation for polyatomic gases in the subsonic part of the nozzle. In connection with thrust calculations for microthruster nozzles, Rae[5] presented a scheme for the numerical solution of slender channel equations with wall-slip boundary conditions for frozen flow. This scheme has been modified and used to calculate flowfields in low Reynolds number hypersonic nozzle where slender channel equations appear to describe the flow adequately.[6,7]

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Governing Equations and Method of Solution

Slender channel equations for two-dimensional and axisymmetric frozen flow have been presented in Refs. 4-7. The equations are formally identical to boundary-layer equations except that the pressure gradient dp/dz along the channel is not known a priori. We present here only a vibrational rate equation, which we obtained by subjecting a modified form of rate equation of Lunkin[8] which takes care of diffusion to slender channel ordering:

$$\rho k \frac{\partial e}{\partial z} + \rho v \frac{\partial e}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu r}{Pr_v} \frac{\partial e}{\partial r} \right) + \rho \frac{e_* - e}{\tau} \tag{1}$$

Here e and e_* are vibrational and equilibrium vibrational energies, z and r are axial and normal coordinates, u and v are axial and normal velocities, and ρ and τ are density and local relaxation time, respectively. Pr_v is a vibrational Prandtl number defined as $Pr_v = \mu c_{vib}/k_{vib}$, where μ is a viscosity based on translational temperature, and c_{vib} and k_{vib} are vibrational specific heat and heat conductivity, respectively. Reference 8 has shown that $c_{vib}/k_{vib} = 1/\rho D_{ii}$, where D_{ii} is the coefficient of self-diffusion. For simplicity, we have used $Pr_v = 0.72$.

The wall boundary condition of Eq. (1) is difficult. For "wall-slip," a vibrational temperature condition has been used in Ref. 7. For the no-slip case, we assume a boundary condition, as is the practice for dissociated boundary layers with wall recombination:

$$e - e_{*w} = [(\mu/\alpha_v \rho \, Pr_v)\cos\vartheta(\partial e/\partial r)]_w \tag{2}$$

Here ϑ is the wall inclination angle, subscript w stands for wall, and α_v is the wall catalyticity for vibrational duxcitation and has the dimension of velocity. We notice that $\alpha_v \to \infty$ will mean a perfectly catalytic wall and $e_w = e_{*w}$, and $\alpha_v \to 0$ will mean no vibrational heat exchange with the wall. Following Rae,[5] we derived a stream tube relation from the continuity equation by replacing z derivatives of all flow variables in favor of the axial pressure gradient dp/dz by using momentum, energy, rate, and state (perfect gas) equations. This equation is

$$\partial/\partial r[r(v/u)] = B_1(dp/dz) + B_2 \tag{3}$$

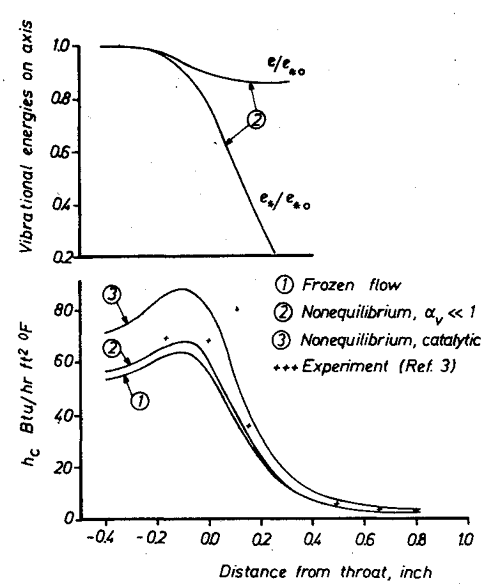

Fig. 1 Calculated vibrational and equilibrium vibrational energies along the nozzle axis and experimental heat-transfer coefficient h_c for nozzle of Ref. 3, run 2.